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A Tractable Multiple Agents Protocol and Algorithm for Resource Allocation under Price Rigidities

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Abstract In many resource allocation problems, economy efficiency must be taken into consideration together with social equality, and price rigidities are often made according to some economic and social needs. We investigate the computational issues of dynamic mechanisms for selling multiple indivisible objects under price rigidities. We propose a multiple agents protocol and algorithm with polynomial time complexity that can achieve the over-demanded sets of items, and then introduce a dynamic mechanism with rationing to discover constrained Walrasian equilibria under price rigidities in polynomial time. We also address the computation of buyers' expected profits and items' expected prices, and discuss strategical issues in the sense of expected profits.

Keywords Resource Allocation · Price Rigidity · Multiple Agents System · Dynamic Mechanism · Constrained Walrasian Equilibrium

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1 Introduction

Problem of allocating resources among selfish agents has been a well-established research theme in economics and recently becomes an emerging research topic because Artifact Intelligence (AI) methodologies can provide computational techniques [25, 26, 31] to the balancing of computation tractability and economic (or societal) needs in these problems [7, 23].

Dynamic mechanisms for resource allocation are trading mechanisms for discovering market-clearing prices and equilibrium allocations based on price adjustment processes [31, 1, 14]. Assume a seller wishes to sell a set of indivisible items to a number of buyers. The seller announces the current prices of the items and the buyers respond by reporting the set of items they wish to buy at the given prices. The seller then calculates the over-demanded set of items and increases the prices of over-demanded items. This iterative process continues until all the selling items can be sold at the prices at which each buyer is assigned with items that maximize her personal net benefit.

Different from one-shot combinatorial auctions [11], the main issue of a dynamic mechanism is whether the procedure can lead to an equilibrium state (Walrasian equilibrium) at which all the selling items are effectively allocated to the buyers (equilibrium allocation) and the price of items gives the buyers their best values [20, 28, 13].

Most of the discussions on the issues of dynamic mechanisms are based on market models in which there does not exist price rigidities. In fact, “good” allocations must look after both sides economy efficiency and social equality, and price rigidities may play a key role in some of these problems. For instance, in an estate bubble period, housing cost is unbearable for most of the members of society. The government may need to allocate some housing resources (whose prices are not completely flexible but restricted under some price rigidities) to middle-income earners. On one hand, the lower bound prices can be made according to some basic economic requirements (e.g., construction costs); on the other hand, the upper bound prices[1] should be made according to some realistic social foundation (e.g., average income level or pay ability). It is well-known that a Walarasian equilibrium exists in the economy when there are no price rigidities. In the case of price restrictions, a Walrasian equilibrium may not exist since the equilibrium price vector may not be admissible.

Talman and Yang studied the equilibrium allocation of heterogeneous indivisible items under price rigidities, and proposed the concept of constrained Walrasian equilibria [29]. A constrained Walrasian equilibrium consists of a price vector $\mathbf{p}$, a rationing system R, and a (constrained) equilibrium allocation π [20] s.t. $\mathbf{p}$ obeys the price rigidities, and π assigns each buyer an item (permitted by R) that maximizes her personal net benefit at $\mathbf{p}$. They also proposed two dynamic auction procedures that produce constrained Walrasian equilibria. However, the computational issues of these procedures have not been touched.

In this paper, we present a polynomial algorithm that can be used to find over-demanded sets of items, and then introduce a dynamic mechanism (multiple agents price rigidities, abbr. MAPR) with rationing to discover constrained Walrasian equilibria under price rigidities in polynomial time. In MAPR, buyers compete with each other (with the help of the seller) on prices of items for multiple rounds. In each round, the seller announces the current price vector (initially, the lower bound price vector) of the items that remain, then the buyers respond by reporting the set of resources they wish to buy, then the seller computes a minimal over-demanded set X_{min} of the items. If $X_{min} = \emptyset$ then the final allocation is computed by the RM (Renewing-Matching) subroutine and MAPR stops. Otherwise if all the prices of the items in X_{min} are less than their upper bounds then the seller increases them; else an item $a \in X_{min}$ (whose price is on its upper bound) is picked and the buyers who only demand some items (including a) in X_{min} draw lots for the right to buy a. Since MAPR's execution process is nondeterministic, we define the concepts of buyers' expected profits and items' expected prices, and consider strategical issues (in the sense of expected profit) in MAPR.

[1] Note that since upper bound prices are often set for the sake of equality between social members (who have some but limited pay ability), they generally accompany a limit to the number of resources one member can get.

Here are main contributions of our work:

- We address the computational problems of dynamic auction proposed by Talman and Yang [29], where these problems have not been touched.
- We complete the proof about the existence of constrained Walrasian equilibrium. And a multiple agents under price rigidities algorithm is proposed to obtain the final allocation and several lemmas to prove the criteria required in constrained Walrasian equilibrium.
- We define the “expected profits” and “expected prices” and discuss strategical issues.

The remainder of the paper is organized as follows. Related works about allocating resources and its algorithms are reviewed in Section 2. In Section 3 some basic notions and examples relating to our work are briefly explained, especially the constrained Walrasian equilibrium. Then, we represent demand situations with bipartite graphs. In Section 4, we address the computation of minimal over–demanded sets of items. Then our MAPR algorithm is presented. We prove formally that it yields a constrained Walrasian equilibrium in polynomial time. In Section 5, we consider strategical issues in MAPR. Finally conclusions are given in Section 6.

2 Related Works

The problem of allocating resources among multiple agents relates to multiple disciplines, including computer science, artificial intelligence, microeconomics, computational social choice, psychology [2,23,12]. Approaches to resource allocation can be classified according to three dichotomies: (1) centralized or decentralized approaches; (2) divisible or indivisible objects; (3) allowing money transfers or not. (2) and (3) need no explanation. In centralized approaches (e.g., the approach given in [22], and combinatorial auction discussed in [11]), agents are required to fully reveal their preferences to a central authority, who computes the final allocation. In decentralized approaches (e.g., in [1], [9]), the final allocation is determined (possibly with the help of a central authority) by agents' interactions that reveal only a part of their preferences.

Many centralized approaches to allocating indivisible goods have been proposed. Guo and Deligkas [15] study probabilistic single-item second-price auctions where the item is characterized by a set of attributes. The auctioneer chooses to reveal only a subset of these attributes to the bidders to create thicker market, which may lead to higher revenue. Because designing revenue-maximizing combinatorial auctions is very hard even for two bidders and two items for sale, Likhodedov and Sandholm [21] focus on designing suboptimal auction mechanisms which yield high revenue and propose two approaches to constructing approximately optimal combinatorial auction. In addition to economy efficiency, social equality and fairness are also discussed. Hartline and Yan [16] consider profit maximizing mechanism design in position auctions and single-minded combinatorial auctions and consider the fairness constraint of envy-freeness in addition to incentive compatible.

However, Bouveret and Lang [3] point that centralized approaches have two drawbacks: (a) elicitation process and winner determination algorithm can be very expensive; (b) agents have to reveal their full preferences, which they might be reluctant to do. Ausubel [1] proposes an efficient decentralized approach for auctioning multiple heterogeneous commodities. Ausubel claims that the proposed mechanism yields an efficient allocation as from a Vickrey-Clarke-Groves mechanism, but it offers advantages of simplicity and transparency. More importantly, it maintains privacy in the sense that buyers avoid the need to report full preferences one-shot to the auctioneer. Sun and Yang [28] propose another decentralized approach called a double-track procedure for efficiently allocating multiple heterogeneous indivisible items in two distinct sets to some buyers. In each round of the process, the prices of items in one set increase and those of items in the other set decrease. Zhang et al. [31] address computational issues of this procedure and propose an algorithm that could find a Walrasian equilibrium in polynomial time. Some decentralized approaches to allocating indivisible goods in which money transfers are not allowed, have also been proposed recently. Brams [4] adapted a cake-cutting protocol to the allocation of indivisible goods, and the protocol is typically designed for the cases when there are only two agents. Bouveret and Lang [3] study a sequential elicitation-free protocol. In the protocol, agents take turns to pick their favorite goods according to a designated sequence. Kalinowski et al. study the strategy behavior [18][19] and what is the “best” sequence [17] of sequential elicitation-free protocol.

There are some important research works (e.g., [5], [6], [8], [10]) that discuss how to allocate divisible goods among multiple agents. More information can be seen in [30] and Procaccia's survey [24].

Table 1 Dichotomies of approaches to resource allocation

Allowing money transfers or not	Centralized approaches		Decentralized approaches	
	Divisible objects	Indivisible objects	Divisible objects	Indivisible objects
No money transfer	[5], [8], [10]	[22]	[6], [24]	[3], [4], [9], [17], [18], [19]
Allowing money transfers	[30]	[11], [15], [16], [21]	N/A	[1], [9], [28], [31], [29]* and our work*

Table 1 clusters the related work according to the three dichotomies (i.e., centralized or decentralized approaches, divisible or indivisible objects, and allowing money transfers or not). Note that only [29] and our work consider price rigidities.

3 Problem and Examples

We formulate the problem and provide some examples relevant to our work, especially the constrained Walrasian equilibrium.

3.1 Constrained Walrasian Equilibrium

Consider a market situation where a seller wishes to sell a finite set X of indivisible items to a finite number of buyers $N = \{1, 2, \ldots, n\}$. The item $o \in X$ is a dummy item which can be assigned to more than one buyer. Items (eg., houses or apartments) in $X \setminus \{o\}$ may be heterogeneous.

A price vector $\mathbf{p} \in \mathbb{Z}_+^X$ assigns a non-negative integer to each $a \in X$ and $\mathbf{p}_a$ is the price of a under $\mathbf{p}$. It is required that $\mathbf{p}_a$ is not completely flexible and restricted to an interval $[\underline{\mathbf{p}}_a, \overline{\mathbf{p}}_a]$ s.t. $\underline{\mathbf{p}}_a, \overline{\mathbf{p}}_a \in \mathbb{Z}_+$, $\underline{\mathbf{p}}_a \leq \overline{\mathbf{p}}_a$, and $0 = \underline{\mathbf{p}}_o = \overline{\mathbf{p}}_o$. We say $\underline{\mathbf{p}}$ and $\overline{\mathbf{p}}$ as the lower and upper bound price vectors. $P = \{\mathbf{p} \in \mathbb{Z}_+^X | (\forall a \in X)\ \underline{\mathbf{p}}_a \leq \mathbf{p}_a \leq \overline{\mathbf{p}}_a\}$ is called the set of *admissible* price vectors. Each $i \in N$ has an integer value function, i.e., $u_i : X \to \mathbb{Z}_+$. $u_i(a)$ is i's valuation to item a. We assume u_i is i's private information, $u_i(o) = 0$, and i can pay $\max_{a \in X} \overline{\mathbf{p}}_a$ units of money. We say $E = \langle N, X, \{u_i\}_{i \in N} \rangle$ is an *economy*.

A rationing system is a function $R : N \times X \to \{0, 1\}$ s.t. $R(i, o) = 1$ for every $i \in N$. $R(i, a) = 1$ means that buyer i is allowed to demand item a, while $R(i, a) = 0$ means that i is not allowed to demand a. At $\mathbf{p} \in P$ and rationing system R, the indirect utility $V_i(\mathbf{p}, R)$

and constrained demand $D_i(\mathbf{p}, R)$ of buyer i is given by: $V_i(\mathbf{p}, R) = \max\{u_i(a) - \mathbf{p}_a | a \in X$ and $R(i, a) = 1\}$, and $D_i(\mathbf{p}, R) = \{a \in X | R(i, a) = 1$ and $u_i(a) - \mathbf{p}_a = V_i(\mathbf{p}, R)\}$. An *allocation* of X is a function $\pi : N \to X$ s.t. $\pi(i) \neq \pi(j)$ if $j \neq i$ and $\pi(i) \in X \setminus \{o\}$. π is an *equilibrium allocation* if $\pi(i) \in D_i(\mathbf{p}, R)$ for all $i \in N$.

$\langle \mathbf{p}, R, \pi \rangle$ is a *constrained Walrasian equilibrium* if

1. $\mathbf{p} \in P$, R is a rationing system,
2. π is an equilibrium allocation,
3. $\mathbf{p}_a = \underline{\mathbf{p}}_a$ if $\pi(i) \neq a$ for all $i \in N$,
4. $\mathbf{p}_a = \overline{\mathbf{p}}_a$ and $\pi(i) = a$ for some $i \in N$ if $R(j, a) = 0$ for some $j \in N$, and
5. $a \in D_i(\mathbf{p}, R')$ if $R(i, a) = 0$, where $R'(j, b) = R(j, b)$ for all $\langle j, b \rangle \in N \times X$ except $R'(i, a) = 1$.

Conditions 1 and 2 need no explanation. Condition 3 says that if the price of an item is greater than its lower bound then it must be assigned to some buyer. Condition 4 states that if an buyer is not allowed to demand some items then the item must be assigned to another buyer at its upper bound price. Condition 5 says that if an buyer is allowed to demand an item which she was not allowed to demand, then she will demand the item. We claim that *constrained Walrasian equilibrium* is a natural generalization of *Walrasian equilibrium* under price rigidities. All the five conditions above make a balance between efficiency and equality.

The following example is modified from the one given in [29]. It illustrates the notions introduced in this section and will be used throughout the paper.

Example 1 *Let $E = \langle N, X, \{u_i\}_{i \in N} \rangle$ be an economy such that $N = \{1, 2, 3, 4, 5\}$, $X = \{o, a, b, c, d\}$, and buyers' values are given in Table 2; price vector $\boldsymbol{p} = (0, 5, 4, 4, 7)$; and π be an allocation of X such that $\pi(1) = o$, $\pi(2) = c$, $\pi(3) = b$, $\pi(4) = a$, and $\pi(5) = d$. Suppose the lower and upper bound price vectors are $\underline{\boldsymbol{p}} = (0, 5, 4, 1, 5)$, and $\overline{\boldsymbol{p}} = (0, 6, 6, 4, 7)$, respectively. So $\boldsymbol{p}$ is an admissible price vector. Let R be a rationing system such that $R(i, x) = 1$ for all $\langle i, x \rangle \in N \times X$ except that $R(3, c) = R(1, c) = 0$. For each buyer $i \in N$, $V_i(\boldsymbol{p}, R)$ and $D_i(\boldsymbol{p}, R)$ are also shown in Table 2. Obviously, $\langle \boldsymbol{p}, R, \pi \rangle$ is a constrained Walrasian equilibrium.*

Table 2 Values, indirect utilities, and constrained demand

buyer i	$u_i(o)$	$u_i(a)$	$u_i(b)$	$u_i(c)$	$u_i(d)$	$V_i(\mathbf{p}, R)$	$D_i(\mathbf{p}, R)$
1	0	4	3	5	7	0	{o,d}
2	0	7	6	8	3	4	{c}
3	0	5	5	8	7	1	{b}
4	0	9	4	3	2	4	{a}
5	0	6	2	4	10	3	{d}

3.2 Demand Situation and Maximum Consistent Allocation

Given an economy $E = \langle N, X, \{u_i\}_{i \in N} \rangle$, we call $\mathcal{D} = (D_i)_{i \in N}$ a *demand situation* of E if there is a price vector $\mathbf{p}$ and a rationing system R such that $D_i = D_i(\mathbf{p}, R)$ for all $i \in N$. An allocation π is *consistent* with $\mathcal{D}$ if $\pi(i) \in D_i \cup \{o\}$ for all $i \in N$. π is maximum if $|\{i \in N | o \notin D_i$ *and* $\pi(i) \neq o\}| \geq |\{i \in N | o \notin D_i$ *and* $\pi'(i) \neq o\}|$ for every allocation π' consistent with $\mathcal{D}$.

$\mathcal{D}$ can be represented as a bipartite graph $\mathrm{BG}(\mathcal{D}) = \langle N' \cup X', \mathcal{E} \rangle$ where $N' = \{i \in N | o \notin D_i\}$, $X' = \bigcup_{i \in N'} D_i$, and $\mathcal{E} = \{\{i, a\} | i \in N', a \in D_i\}$. A *matching* in $\mathrm{BG}(\mathcal{D})$ is a subset M of $\mathcal{E}$ s.t. $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$. M is maximum if $|M'| \leq |M|$ for each matching M'.

It is not hard to see that a matching M in $\mathrm{BG}(\mathcal{D})$ determines an allocation consistent with $\mathcal{D}$. π^M denotes the allocation determined by M, that is, $\pi^M(i) = a$ if $\exists \{i, a\} \in M$, and $\pi^M(i) = o$ otherwise. Suppose M is maximum, then π^M is maximum and it is easy to find that: there exists an equilibrium allocation $\Leftrightarrow |M| = |\{i \in N | o \notin D_i\}| \Leftrightarrow \pi^M$ is an equilibrium allocation.

In fact, to find a maximum matching in a bipartite graph is a pure combinatorial optimization problem, which can be addressed in polynomial time. Schrijver [27] presents the matching augmenting algorithm MA, which takes a bipartite graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ and a matching M in $\mathcal{G}$ as input, and outputs a matching $\mathrm{MA}(\mathcal{G}, M) = M'$ s.t. $|M'| \geq |M|$ and $\bigcup_{e \in M'} e \supseteq \bigcup_{e \in M} e$ in time $O(|\mathcal{E}|)$. So a maximum matching can be found in time $O(|\mathcal{V}||\mathcal{E}|)$ (as we do at most $|\mathcal{V}|$ iterations), i.e., $O(|N||X| \min(|N|, |X|))$. In the following discussion, $\hat{M}_{\mathcal{D}}$ denotes the maximum matching of $\mathrm{BG}(\mathcal{D})$ found by this way.

Example 2 *See the economy given in Example 1. Let price vector $\boldsymbol{p} = (0, 5, 4, 3, 5)$ and R be the rationing system such that $R(i, a) = 1$ for all $\langle i, a \rangle \in N \times X$. Then buyers' constrained demands at $\boldsymbol{p}$ and R are: $D_1(\boldsymbol{p}, R) = \{c, d\}$, $D_2(\boldsymbol{p}, R) = D_3(\boldsymbol{p}, R) = \{c\}$, $D_4(\boldsymbol{p}, R) = \{a\}$, $D_5(\boldsymbol{p}, R) = \{d\}$. Let $\mathcal{D} = (D_i(\boldsymbol{p}, R))_{i \in N}$. $\hat{M}_{\mathcal{D}} = \{\{1, c\}, \{4, a\}, \{5, d\}\}$.*

4 Mechanism for Allocating Resources under Price Rigidities

4.1 Over-demanded set of items

What can lead to non-existence of equilibrium allocations? This is a key issue that we need to consider.

Given a demand situation $\mathcal{D} = (D_i)_{i \in N}$, a set of real items $X' \subseteq X \setminus \{o\}$ is *over-demanded* in $\mathcal{D}$, if the number of buyers who demand only items in X' is strictly greater than the number of items in X', i.e., $|\{i \in N | D_i \subseteq X'\}| > |X'|$; X' is *not under-demanded*, if the number of buyers who demand some items in X' is not less than the number of items in X', i.e., $|\{i \in N | D_i \cap X' \neq \emptyset\}| \geq |X'|$. An over-demanded set X' is *minimal* if no strict subset of X' is over-demanded. We can get Lemma 1 directly based on these definitions.

Lemma 1 *Let $X' \subseteq X \setminus \{o\}$ is over-demanded. Then for each $a \in X'$, either there exists a minimal over-demanded set $X'' \subseteq X'$ s.t. $a \notin X''$, or $a \in X''$ for every minimal over-demanded set $X'' \subseteq X'$.*

Theorem 1 answers the question proposed in the beginning of this section.

Theorem 1 *There exists an over-demanded set of items in $\mathcal{D} = (D_i)_{i \in N}$ if and only if there does not exist an equilibrium allocation.*

Proof Sufficiency is obvious. Let us prove necessity. Suppose there does not exist an equilibrium allocation. Let $M = \hat{M}_{\mathcal{D}}$ and $N' = \{i \in N | o \notin D_i\}$. Then $|M| = |N \cap \bigcup_{e \in M} e| < |N'|$. Pick a buyer i from $N' \setminus N \cap \bigcup_{e \in M} e$. We construct a sequence $\langle X_0, N_0 \rangle, \langle X_1, N_1 \rangle, \cdots$ as follows:

- $X_0 = D_i$, $N_0 = \{j \in N | (\exists a \in X_0)\{j, a\} \in M\}$;
- $X_{k+1} = \bigcup_{j \in N_k} D_j$; and $N_{k+1} = \{j \in N | (\exists a \in X_{k+1})\ \{j, a\} \in M\}$.

Pick any $k \geq 0$ and $a \in X_k$. Suppose there does not exist $j \in N$ such that $\{j, a\} \in M$. Then there is an M-augmenting path [27] from a to i, i.e., M is not maximum, contradicting the fact that M is maximum. So for all $k \geq 0$ and $a \in X_k$, there exists $j \in N$ such that $\{j, a\} \in M$. Consequently,

1. $X_k \subseteq X_{k+1} \subseteq X$, $N_k \subseteq N_{k+1} \subseteq N$ for all $k \geq 0$;
2. if $X_{k+1} = X_k$ then $X_{k+l} = X_k$ and $N_{k+l} = N_k$ for all $k, l \geq 0$.

So there must exist $K \geq 0$ s.t. $X_0 \subset \cdots \subset X_K = X_{K+1} = \cdots$. For each $b \in X_K$, b is assigned to only one buyer in N_K at π^M. And for each $j \in N_K$, $D_j \subseteq X_K$ and j is assigned with only one item in X_K at π^M. So $|X_K| = |N_K|$. Consequently, $|\{i \in N | D_i \subseteq X_K\}| \geq |N_K \cup \{i\}| = |N_K| + 1 = |X_K| + 1 > |X_K|$. So X_K is an over-demanded set of items in $\mathcal{D}$.

To find a minimal over-demanded set of items, we develop the MODS algorithm shown in Algorithm 1. Given a demand situation $\mathcal{D}$, and $\hat{M}_{\mathcal{D}}$ s.t. $|\hat{M}_{\mathcal{D}}| < |\{i \in N | o \notin D_i\}|$, MODS returns a minimal over-demanded set of items X_{min}. The basic idea of MODS is to generate an over-demanded set X' firstly (see lines 3–8 in Algorithm 1), and then (according to Lemma 1) to find a minimal over-demanded set $X_{min} \subseteq X'$ (see lines 10–18 in Algorithm 1).

ALGORITHM 1: MODS (minimal over-demanded set) algorithm

Input: $(D_i)_{i \in N}$, $\hat{M}_{\mathcal{D}}$
Output: X_{min}
1 $\mathcal{D} \leftarrow (D_i)_{i \in N}$;
2 $M \leftarrow \hat{M}_{\mathcal{D}}$;
3 Pick i from $\{i \in N | o \notin D_i\} \setminus \bigcup_{e \in M} e$;
4 $X'' \leftarrow D_i$, $X' \leftarrow \emptyset$;
5 **while** $(X'' \neq \emptyset)$ **do**
6 $\quad N' \leftarrow \{j \in N | (\exists a \in X'')\{j, a\} \in M\}$;
7 $\quad X' \leftarrow X' \cup X''$;
8 $\quad X'' \leftarrow \bigcup_{j \in N'} D_j \setminus X'$;
9 **end**
10 $X_{min} \leftarrow \emptyset$, $X'' \leftarrow X'$;
11 **for** $\forall a \in X'$ **do**
12 $\quad X'' \leftarrow X'' \setminus \{a\}$;
13 $\quad N' \leftarrow \{i \in N | D_i \subseteq X_{min} \cup X''\}$;
14 $\quad \mathcal{D}' \leftarrow (D_i)_{i \in N'}, k \leftarrow |\hat{M}_{\mathcal{D}'}|$;
15 $\quad$ **if** $k = |N'|$ **then**
16 $\quad\quad X_{min} \leftarrow X_{min} \cup \{a\}$;
17 $\quad$ **end**
18 **end**
19 **return** X_{min};

The correctness of algorithm MODS is directly from Lemma 1 and the proof of Theorem 1. Let $BG(\mathcal{D}) = \langle \mathcal{V}, \mathcal{E} \rangle$. We can find easily the facts as follows.

1. In order to generate an over-demanded set X' (lines 3–8 in Algorithm 1), MODS only visits edges in $\mathcal{E}$. For each $e \in \mathcal{E}$, e can be visited once at most.
2. $|X'| \leq |\hat{M}_{\mathcal{D}}| \leq \min(|N|, |X|)$, and $BG(\mathcal{D}') \subseteq BG(\mathcal{D})$ (see line 14).

Because $|\mathcal{E}| \leq |N||X|$ and the complexity of $\hat{M}_{\mathcal{D}}$ is in $O(|N||X| \min(|N|, |X|))$, the overall complexity of MODS $(\mathcal{D}, \hat{M}_{\mathcal{D}})$ is in $O(|N||X|(\min(|N|, |X|))^2)$.

Example 3 *See $\mathcal{D}$ and $\hat{M}_{\mathcal{D}}$ described in Example 2. It is easy to find that $|\hat{M}_{\mathcal{D}}| < |\{i \in N | o \notin D_i\}|$. We apply* MODS *algorithm to $(\mathcal{D}, \hat{M}_{\mathcal{D}})$. Figure 1 and Figure 2 provide a graphical illustration of the application. $\mathcal{D}$ is represented as a bipartite, and the bold lines denote the maximum matching $\hat{M}_{\mathcal{D}}$. Firstly, an over-demanded set $X' = \{c, d\}$ is found. And then a minimal over-demanded set $X_{min} = \{c\}$ is found.*

4.2 MAPR Algorithm

In this subsection, we present a polynomial mechanism for resource allocation under price rigidities (MAPR).

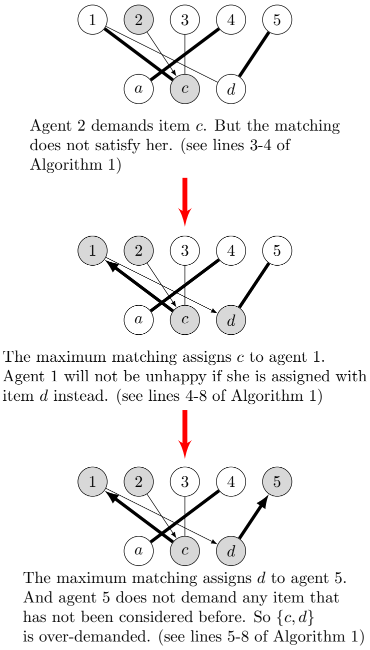

Fig. 1 How does MODS find an over-demanded set

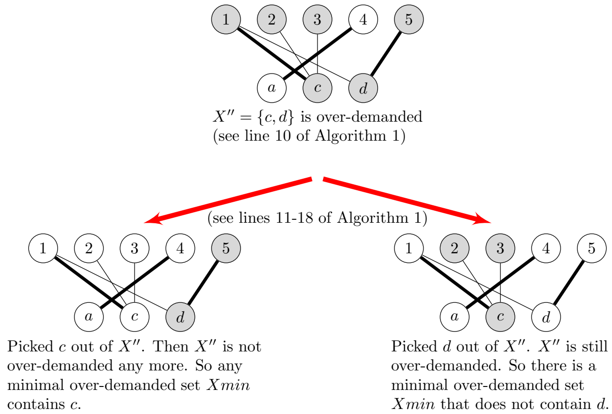

Fig. 2 How does MODS find a minimal over-demanded set

ALGORITHM 2: MAPR algorithm

1. The seller φ announces the set X of items to allocate, and sets $\mathbf{p}^0 \leftarrow \underline{\mathbf{p}}$, $M^0 \leftarrow \emptyset, N' \leftarrow N$. Each buyer $i \in N$ sets $R_i[a] \leftarrow 1$ for each item $a \in X$. Let $t \leftarrow 0$.
2. φ sends $\mathbf{p}^t$ and "Report your demand." to each $i \in N'$.
3. Each $i \in N'$ computes and sends $D_i = \{a \in X | R_i[a] = 1 \text{ and } u_i(a) - \mathbf{p}^t_a = \max\{u_i(b) - \mathbf{p}^t_b | R_i[b] = 1\}\}$ to φ.
4. φ computes $N'' = \{i \in N' | D_i \cap \bigcup_{e \in M^t} e \neq \emptyset\}$. If $N'' = \emptyset$ then go to Step (6). φ sends "Sorry, items in $D'_i = D_i \cap \bigcup_{e \in M^t} e$ have been sold. Please report your new demand." to each $i \in N''$, and sets $N' \leftarrow N''$.
5. Each $i \in N'$ sets $R_i[a] \leftarrow 0$ for all $a \in D'_i$. Go to Step (3).
6. Let $N^* = N \setminus \bigcup_{e \in M^t} e$ and $\mathcal{D}^* = (D_i)_{i \in N^*}$. φ computes $\hat{M}_{\mathcal{D}^*}$. If $|\hat{M}_{\mathcal{D}^*}| = |\{i \in N^* | o \notin D_i\}|$ then go to Step (9). φ computes $X_{min} = \text{MODS}(\mathcal{D}^*, \hat{M}_{\mathcal{D}^*})$.
7. φ computes $\overline{X} = \{a \in X_{min} | \mathbf{p}^t_a = \overline{\mathbf{p}}_a\}$. If $\overline{X} = \emptyset$ then φ sets $N' \leftarrow N^*$, $M^{t+1} \leftarrow M^t$, $\mathbf{p}^{t+1}_a \leftarrow \mathbf{p}^t_a + 1$ for all $a \in X_{min}$, and $\mathbf{p}^{t+1}_a \leftarrow \mathbf{p}^t_a$ for all $a \in X \setminus X_{min}$. Let $t \leftarrow t + 1$. Go to Step (2).
8. φ picks an item a from $\overline{X}$ and asks the buyers in $\{i \in N^* | a \in D_i \subseteq X_{min}\}$ to draw lots for the right to buy a. Let i be the winning buyer. φ sets $M^{t+1} \leftarrow M^t \cup \{\{i, a\}\}$, $N' \leftarrow N^* \setminus \{i\}$ and $\mathbf{p}^{t+1} \leftarrow \mathbf{p}^t$. Let $t \leftarrow t + 1$. Go to Step (2).
9. φ computes $M^* \leftarrow M^t \cup \text{RM}((D_i)_{i \in N}, M^t, \mathbf{p}^t, \underline{\mathbf{p}})$ and then announces $\mathbf{p}^t$ and π^{M^*} are the final price vector and allocation. MAPR stops.

ALGORITHM 3: RM (Renewing-Matching) algorithm

Input: $(D_i)_{i \in N}, M, \mathbf{p}, \underline{\mathbf{p}}$
Output: $\tilde{M}$
$X' \leftarrow \{a \in X \setminus \bigcup_{e \in M} e | \mathbf{p}_a > \underline{\mathbf{p}}_a\}$;
$N' \leftarrow \{i \in N \setminus \bigcup_{e \in M} e | D_i \cap X' \neq \emptyset\}$;
$\mathcal{D}' \leftarrow (D_i \cap X')_{i \in N'}$;
$M' \leftarrow \hat{M}_{\mathcal{D}'}$;
$N^* \leftarrow N \setminus \bigcup_{e \in M} e$;
$\langle \mathcal{V}, \mathcal{E} \rangle \leftarrow BG((D_i)_{i \in N^*})$;
$M'' \leftarrow M' \cap \mathcal{E}$;
while $(\text{MA}(\langle \mathcal{V}, \mathcal{E} \rangle, M'') \neq M'')$ **do**
    $M'' \leftarrow \text{MA}(\langle \mathcal{V}, \mathcal{E} \rangle, M'')$;
end
$\tilde{M} \leftarrow M'' \cup \{e \in M' | e \cap \bigcup_{e' \in M''} e' = \emptyset\}$;
return $\tilde{M}$;

Its basic idea is to eliminate over-demanded sets of items by increasing the prices of over-demanded items or rationing an over-demanded item whose price has reached its upper bound.

Talman and Yang [29] provide two dynamic procedures that produce constrained Walrasian equilibrium. But it does not address the computation issues, and the third condition of constrained Walrasian equilibrium cannot be guaranteed either. In order to make sure that all the items whose prices exceed their lower bound prices will be sold (the third criterion of constrained Walrasian equilibrium), the RM subroutine shown in Algorithm 3 is called in Step 9 of Algorithm 2. Given a demand situation $\mathcal{D} = (D_i)_{i \in N}$, a partial matching M consistent with $\mathcal{D}$, the current price vector $\mathbf{p}$, and the lower bound price vector $\underline{\mathbf{p}}$, RM returns a matching M' such that **(1)** $\pi^{M \cup M'}$ is an equilibrium allocation, **(2)** $M \cap M' = \emptyset$, and **(3)** $\{a \in X \setminus \bigcup_{e \in M} e | \mathbf{p}_a > \underline{\mathbf{p}}_a\} \subseteq \bigcup_{e \in M'} e$.

The rough framework of MAPR is illustrated in Figure 3. Observing MAPR and RM subroutines, we can find that:

– computation of each step is polynomial in $|N|$ and $|X|$;
– for each $t \geq 0$, the number of the loops consisting of Steps 3–5 in Algorithm 2 is not more than $|X|$; and

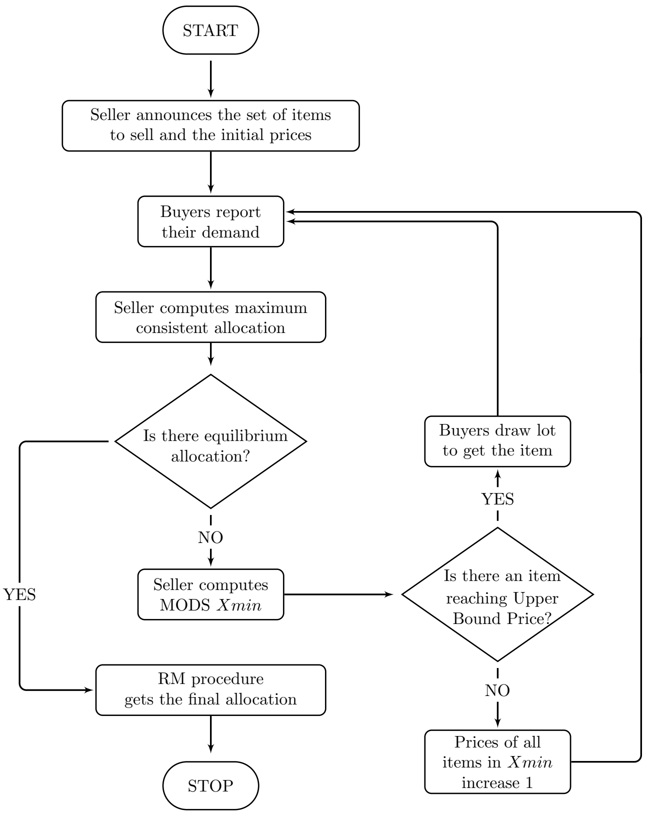

Fig. 3 Framework of Algorithm MAPR

– the number of the loops consisting of Steps 2–8 is not more than $\sum_{a\in X}(\overline{\mathbf{p}}_a - \underline{\mathbf{p}}_a)$.

So MAPR always terminates and is polynomial in $|N|$, $|X|$, and $\sum_{a\in X}(\overline{\mathbf{p}}_a - \underline{\mathbf{p}}_a)$.

In order to prove the correctness of MAPR and RM, we will first give some definitions and provide three lemmas, then we will prove that MAPR can lead to a constrained Walrasian equilibrium with the help of these three lemmas. In the following discussion, we suppose that MAPR terminates at some time $T \geq 0$; $\mathbf{p}^t$, M^t, R^t ($R^t(i,a) = R_i[a]$ for all $\langle i,a\rangle \in N \times X$, where R_i is the vector kept by buyer i at time t), and $(D_i^t)_{i\in N}$ denote the price vector, partial matching that has been made so far, rationing system, and demand situation at time $0 \leq t \leq T$, respectively. Let $X^t = \{a \in X \setminus \bigcup_{e\in M^t} e | \mathbf{p}_a^t > \underline{\mathbf{p}}_a\}$ and $N^t = \{i \in N \setminus \bigcup_{e\in M^t} e | D_i^t \cap X^t \neq \emptyset\}$.

The proof of convergence to a constrained Walrasian equilibrium is not trivial. We have the aid of three auxiliary lemmas (in which $\mathcal{D} = (D_i)_{i\in N}$ denotes a demand situation) to get the proof. These three lemmas are closely connected. The proof of Lemma 4 is based on Lemmas 2 and 3, and the proof of Theorem 2 is based on these three lemmas.

Lemma 2 states that, each nonempty subset of a minimal over-demanded set of items is not under-demanded.

Lemma 2 *Let X' be a minimal over-demanded set of items. Then for each $\emptyset \subset X'' \subseteq X'$, $|\{i \in N | D_i \cap X'' \neq \emptyset$ and $D_i \subseteq X'\}| > |X''|$.*

The proof of Lemma 2 is not very hard, and comes from using the reduction to absurdity.

Proof Suppose there exists $\emptyset \subset X'' \subseteq X'$ such that

$$|\{i \in N | D_i \cap X'' \neq \emptyset \text{ and } D_i \subseteq X'\}| \leq |X''| \quad (1)$$

Because X' is an over-demanded set of items, we have $|\{i \in N | D_i \subseteq X'\}| > |X'|$. So $|\{i \in N | D_i \subseteq X' \setminus X''\}| + |\{i \in N | D_i \cap X'' \neq \emptyset \text{ and } D_i \subseteq X'\}| > |X''| + |X' \setminus X''|$. According to equation (1), we have $|\{i \in N | D_i \subseteq X' \setminus X''\}| > |X' \setminus X''|$. So $X' \setminus X''$ is an over-demanded set of items, contradicting the fact that X' is a minimal over-demanded set of items. Consequently, for each $\emptyset \subset X'' \subseteq X'$, $|\{i \in N | D_i \cap X'' \neq \emptyset \text{ and } D_i \subseteq X'\}| > |X''|$.

Lemma 3 states that, the cardinality of a maximum matching is not less than the cardinality of a set of real items if each subset of the set is not under-demanded.

Lemma 3 *Let $X' \subseteq X \setminus \{o\}$ and $|\{i \in N | D_i \cap X'' \neq \emptyset\}| \geq |X''|$ for each $X'' \subseteq X'$. If M is a maximum matching of $BG((D_i \setminus \{o\})_{i\in N})$, then $|M| \geq |X'|$.*

The proof of Lemma 3 is similar to that of Theorem 1.

Proof Let $N' = \{i \in N | D_i \cap X' \neq \emptyset\}$ and $M' = \text{MMatching}((D_i \cap X')_{i\in N'})$. According to the correctness of the MMatching algorithm, M' is a maximum matching of the NX graph of $(D_i \cap X')_{i\in N'}$. It is easy to find that $|M| \geq |M'|$ and $|M'| \leq |X'|$.

Suppose $|M'| < |X'|$. Then there exists an item $a \in X' \setminus \bigcup_{e\in M'} e$. We construct a sequence $\langle X_0, N_0\rangle, \langle X_1, N_1\rangle, \ldots$ as follow:

– $X_0 = \{a\}$, $N_0 = \{i \in N' | a \in D_i\}$;
– $X_{k+1} = \{b \in X' | (\exists j \in N_k)\{j,b\} \in M'\}$; and
– $N_{k+1} = \{i \in N | D_i \cap X_{k+1} \neq \emptyset\}$.

Because $|\{i \in N | D_i \cap X'' \neq \emptyset\}| \geq |X''|$ for each $X'' \subseteq X'$, we have $|N_k| \geq |X_k|$ for each $k \geq 0$. Pick any $k \geq 0$ and $i \in N_k$ (if $N_k \neq \emptyset$). Suppose there does not exist $b \in X'$ such that $\{i,b\} \in M'$. Then there is an M-augmenting path from a to i. According to Theorem 1, M' is not maximum, contradicting the fact that M' is maximum. So for all $k \geq 0$ and $i \in N_k$, there exists $b \in X'$ such that $\{i,b\} \in M'$. Consequently, we can find that:

1. $X_k \subseteq X_{k+1} \subseteq X'$, $N_0 \subseteq N_k \subseteq N_{k+1} \subseteq N'$ for all $k > 0$;

2. if $N_{k+1} = N_k$ then $N_{k+l} = N_k$ and $X_{k+1+l} = X_{k+1}$ for all $k, l \geq 0$.

So there must exist $K \geq 0$ such that $N_0 \subset \cdots \subset N_{K-1} \subset N_K = N_{K+1} = \cdots$. For each $i \in N_K$, i is assigned with a distinct item in X_{K+1} at $\pi^{M'}$. And according to X_{K+1}'s definition, we have $|N_K| = |X_{K+1}| = |N_{K+1}|$. Consequently, $|X_0 \cup X_{K+1}| = 1 + |X_{K+1}| > |N_K| = |\{i \in N | D_i \cap (X_0 \cup X_{K+1}) \neq \emptyset\}|$. This contradicts the fact that $|\{i \in N | D_i \cap X'' \neq \emptyset\}| \geq |X''|$ for each $X'' \subseteq X'$. So $|M| \geq |M'| = |X'|$.

Lemma 4 states that, all the items in X^t can be sold.

Lemma 4 *Let $\mathcal{D}^t = (D_i^t \cap X^t)_{i \in N^t}$. Then $|\hat{M}_{\mathcal{D}^t}| = |X^t|$ for each $0 \leq t \leq T$.*

The proof of Lemma 4 is based on Lemmas 2 and 3.

Proof We first prove that $|\{i \in N^t | D_i^t \cap X' \neq \emptyset\}| \geq |X'|$ for each $\emptyset \subset X' \subseteq X^t$ and $0 \leq t \leq T$:

1. It holds at $t = 0$ because $X^0 = \emptyset$.
2. Suppose MAPR does not stop at $\hat{t} \geq 0$ and $|\{i \in N^t | D_i^t \cap X' \neq \emptyset\}| \geq |X'|$ for each $\emptyset \subset X' \subseteq X^t$ and $0 \leq t \leq \hat{t}$.
3. Then $X_{min} \neq \emptyset$ and $\overline{X}$ are computed at time $\hat{t}$ and Steps 6–7 in Algorithm 2. Pick any $\emptyset \subset X' \subseteq X^{\hat{t}+1}$. Let $N_1 = \{i \in N^{\hat{t}} | D_i^{\hat{t}} \subseteq X_{min}$ and $D_i^{\hat{t}} \cap X' \neq \emptyset\}$ and $N_2 = \{i \in N^{\hat{t}} | D_i^{\hat{t}} \cap (X' \setminus X_{min}) \neq \emptyset\}$. There are two possibilities:

Case I : $\overline{X} = \emptyset$. So $X^{\hat{t}+1} = X^{\hat{t}} \cup X_{min}$. According to Lemma 2 and Item 2, we have $|N_1| > |X' \cap X_{min}|$ and $|N_2| \geq |X' \setminus X_{min}|$. It is easy to find that $D_i^{\hat{t}+1} \cap X' \neq \emptyset$ for each $i \in N_1 \cup N_2 \subseteq N^{\hat{t}+1}$ and $N_1 \cap N_2 = \emptyset$. So $|\{i \in N^{\hat{t}+1} | D_i^{\hat{t}+1} \cap X' \neq \emptyset\}| \geq |N_1 \cup N_2| = |N_1| + |N_2| > |X' \cap X_{min}| + |X' \setminus X_{min}| = |X'|$.

Case II : $\overline{X} \neq \emptyset$ and some $a \in \overline{X}$ is assigned to some buyer j such that $a \in D_j^{\hat{t}} \subseteq X_{min}$. So $X^{\hat{t}+1} = X^{\hat{t}} \setminus \{a\}$. According to Lemma 2 and Item 2, we have $|N_1| > |X' \cap X_{min}|$ and $|N_2| \geq |X' \setminus X_{min}|$. It is easy to find that $D_i^{\hat{t}+1} \cap X' \neq \emptyset$ for each $i \in (N_1 \setminus \{j\}) \cup N_2 \subseteq N^{\hat{t}+1}$ and $N_1 \cap N_2 = \emptyset$. Consequently, $|\{i \in N^{\hat{t}+1} | D_i^{\hat{t}+1} \cap X' \neq \emptyset\}| \geq |(N_1 \setminus \{j\}) \cup N_2| \geq |N_1| - 1 + |N_2| \geq |X' \cap X_{min}| + |X' \setminus X_{min}| = |X'|$.

Consequently, $|\{i \in N^{\hat{t}+1} | D_i^{\hat{t}+1} \cap X' \neq \emptyset\}| \geq |X'|$.

According to items 1–3, $|\{i \in N^t | D_i^t \cap X' \neq \emptyset\}| \geq |X'|$ for each $X' \subseteq X^t$ and $0 \leq t \leq T$. It is easy to find that $|\hat{M}_{\mathcal{D}^t}| \leq |X^t|$ for each $0 \leq t \leq T$. According to Lemma 3, we have $|\hat{M}_{\mathcal{D}^t}| \geq |X^t|$ for each $0 \leq t \leq T$. So $|\hat{M}_{\mathcal{D}^t}| = |X^t|$ for each $0 \leq t \leq T$.

Now we are ready to establish the following correctness theorem for MAPR (and RM subroutine).

Theorem 2 *If $\langle \mathbf{p}^T, R^T, \pi^{M^T} \rangle$ is found by MAPR, then it is a constrained Walrasian equilibrium.*

Proof $\langle \mathbf{p}^T, R^T, \pi^{M^T} \rangle$ is a constrained Walrasian equilibrium if and only if it satisfies the five conditions shown in Conditions 1–5.

1. Obviously, $\mathbf{p}^T$ is an admissible price vector and R^T is a rationing system.
2. For each buyer i and the item assigned to her $a = \pi^{M^T}(i)$, there are two possibilities: Case I (Step 8 in Algorithm 2), i is the winner of a lottery on item a at some time $T' \leq T$, and Case II (Step 6 and 9 in Algorithm 2), a is assigned to i at time T.
   (a) In case I, $a \in D_i(\mathbf{p}^{T'}, R^{T'})$. So $u_i(a) - \mathbf{p}_a^{T'} \geq u_i(b) - \mathbf{p}_b^{T'}$ for all $b \in \{b \in X | R^{T'}(i, b) = 1\}$. Because $R^{T'}(i, a) = R^T(i, a) = 1$, $\mathbf{p}_a^{T'} = \mathbf{p}_a^T$, $R^{T'}(i, b) \geq R^T(i, b)$ and $\mathbf{p}_b^{T'} \leq \mathbf{p}_b^T$ for all $b \in X$, $u_i(a) - \mathbf{p}_a^T \geq u_i(b) - \mathbf{p}_b^T$ for all $b \in \{b \in X | R^T(i, b) = 1\}$. So $a \in D_i(\mathbf{p}^T, R^T)$.
   (b) In case II, according to the definition of π^{M^T} (see RM subroutine and Steps 6–9), we have $a \in D_i(\mathbf{p}^T, R^T)$.

   Consequently, π^{M^T} is an equilibrium allocation.
3. According to Lemma 4, all the items in X^T are sold. Consequently, $\mathbf{p}_a^T = \underline{\mathbf{p}}_a$ for each $a \in \{b \in X | (\forall i \in N) \pi^{M^T}(i) \neq b\}$. The correctness of RM subroutine can derive from Items 2 and 3 directly.
4. Obviously (see Steps 2–5 and 8 of MAPR mechanism in Algorithm 2), $\mathbf{p}_a^T = \overline{\mathbf{p}}_a$ and $\pi^{M^T}(i) = a$ for some $i \in N$ if $R^T(j, a) = 0$ for some $j \in N$.
5. If $R^T(i, a) = 0$ then i is a loser of a lottery on item a at some time $T' \leq T$. So $u_i(a) - \mathbf{p}_a^{T'} \geq u_i(b) - \mathbf{p}_b^{T'}$ for all $b \in \{b \in X | R^{T'}(i, b) = 1\}$. Because $\mathbf{p}_a^{T'} = \mathbf{p}_a^T$, $R^{T'}(i, b) \geq R^T(i, b)$ and $\mathbf{p}_b^{T'} \leq \mathbf{p}_b^T$ for all $b \in X$, $u_i(a) - \mathbf{p}_a^T \geq u_i(b) - \mathbf{p}_b^T$ for all $b \in \{b \in X | R^T(i, b) = 1\}$. So $a \in D_i(\mathbf{p}^T, R')$ where $R'(j, b) = R^T(j, b)$ for all $\langle j, b \rangle \in N \times X$ except $R'(i, a) = 1$.

So $\langle \mathbf{p}^T, R^T, \pi^{M^T} \rangle$ is a constrained Walrasian equilibrium.

Example 4 *See Example 1. Apply MAPR to $\langle E, \underline{\mathbf{p}}, \overline{\mathbf{p}} \rangle$. The demands, price vectors, rationing system and other relevant data generated by MAPR are illustrated in Table 3, where U_i, D_i, X', N', and X_{min} denote $\{a \in X | R^t(i, a) = 0\}$, $D_i(\mathbf{p}^t, R^t)$, $X \cap \bigcup_{e \in M^t} e$, $N \cap \bigcup_{e \in M^t} e$, and the value of X_{min} computed by the seller at Step 6 and time t. Figure 4 illustrates the execution of steps 1–7 of MAPR algorithm. Figure 5 illustrates the execution of RM algorithm (see step 8 of MAPR algorithm). In Figure 4 and Figure 5, $(D_i)_{i \in N}$ is represented as a bipartite, items in X_{min} and buyers in $\{i \in N | D_i \subseteq X_{min}\}$ are highlighted with gray circles, and the bold* *lines denote M, i.e., the partial matching that have been settled.*

At $t = 3$, the price of c has reached its upper bound 4. The seller assigns randomly c to buyer 2 or buyer 3. So there are two different possible histories of resource allocation from $t = 3$. Along the history of $t = 4.1; 5.1; 6.1$, MAPR finds $\langle \boldsymbol{p}^{6.1}, R^{6.1}, \pi^{M^{6.1}} \rangle$, where $\pi^{M^{6.1}}(1) = o$, $\pi^{M^{6.1}}(2) = c$, $\pi^{M^{6.1}}(3) = b$, $\pi^{M^{6.1}}(4) = a$, and $\pi^{M^{6.1}}(5) = d$. Along the history of $t = 4.2; 5.2; 6.2$, MAPR finds $\langle \boldsymbol{p}^{6.2}, R^{6.2}, \pi^{M^{6.2}} \rangle$, where $\pi^{M^{6.2}}(1) = o$, $\pi^{M^{6.2}}(2) = b$, $\pi^{M^{6.2}}(3) = c$, $\pi^{M^{6.2}}(4) = a$, and $\pi^{M^{6.2}}(5) = d$. It can be found that in both cases, MAPR discovers a constrained Walrasian equilibrium.

Now we compare with two other allocation mechanisms that provide insight into the MAPR's balance between economy efficiency and social equality.

The English Auction (EA) mechanism allocates each item a to the agent i who prefers it most breaking ties at random (i.e., $u_i(a) = \max\{u_j(a) | j \in N\}$), and sets $\boldsymbol{p}_a$ the price of a as:

$$\boldsymbol{p}_a = \begin{cases} \boldsymbol{u} + 1 & \textit{if } \boldsymbol{u} \neq u_i(a) \\ u_i(a) & \textit{otherwise} \end{cases}$$

where $\boldsymbol{u} = \max\{u_j(a) | j \in N \setminus \{i\}\}$. Apply EA to the economy given in Example 1. Then the allocation and the price vector determined by EA could be π_{EA} and $\boldsymbol{p}_{EA}$ such that $\pi_{EA}(1) = \{o\}$, $\pi_{EA}(2) = \{b, c\}$, $\pi_{EA}(3) = \{o\}$, $\pi_{EA}(4) = \{a\}$, $\pi_{EA}(5) = \{d\}$, $\boldsymbol{p}_{EA} = (0, 8, 6, 8, 8)$. Obviously, π_{EA} is more efficient than $\pi^{M^{6.1}}$ (and $\pi^{M^{6.2}}$) because $\sum_{i \in N} \sum_{a \in \pi_{EA}(i)} u_i(a) = \sum_{i \in N} \sum_{a \in \pi^{M^{6.2}}(i)} u_i(a) = 33 > 32 = \sum_{i \in N} \sum_{a \in \pi^{M^{6.1}}(i)} u_i(a)$. However, it seems to be not fair that π_{EA} assigns two real items to agent 2 and does not assign any real item to agent 1 and agent 3.

In the Sequentially Picking (SP) mechanism , agents take turns to pick items according to the sequence $1, 2, 3, \ldots, n$ (every time an agent is designated, she picks one real item out of those that remain), and $\boldsymbol{p}_a$ the price of a is fixed to be $\underline{\boldsymbol{p}}_a$ for each $a \in X$. Apply SP to the economy given in Example 1. Then the allocation and the price vector determined by SP are π_{SP} and $\boldsymbol{p}_{SP}$ such that $\pi_{SP}(1) = \{d\}$, $\pi_{SP}(2) = \{c\}$, $\pi_{SP}(3) = \{a\}$, $\pi_{SP}(4) = \{b\}$, $\pi_{SP}(5) = \{o\}$, $\boldsymbol{p}_{SP} = \underline{\boldsymbol{p}}$. SP guarantees that there are $\min(|N|, |X| - 1)$ agents assigned with real items and each agent is assigned with at most one real item. However, the efficiency of SP can be very poor. For example, $\sum_{i \in N} \sum_{a \in \pi_{SP}(i)} u_i(a) = 24 < 32 = \sum_{i \in N} \sum_{a \in \pi^{M^{6.1}}(i)} u_i(a) < 33 = \sum_{i \in N} \sum_{a \in \pi^{M^{6.2}}(i)} u_i(a)$.

Table 3 Results generated by MAPR

t	$\mathbf{p}_o^t$	$\mathbf{p}_a^t$	$\mathbf{p}_b^t$	$\mathbf{p}_c^t$	$\mathbf{p}_d^t$	X_{min}
0	0	5	4	1	5	$\{c\}$
1	0	5	4	2	5	$\{c\}$
2	0	5	4	3	5	$\{c\}$
3	0	5	4	4	5	$\{c\}$
4.1	0	5	4	4	5	$\{d\}$
5.1	0	5	4	4	6	$\{d\}$
6.1	0	5	4	4	7	$\emptyset$
4.2	0	5	4	4	5	$\{d\}$
5.2	0	5	4	4	6	$\{d\}$
6.2	0	5	4	4	7	$\emptyset$

t	U_1	U_2	U_3	U_4	U_5	N'
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4.1	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\{2\}$
5.1	$\{c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\{2\}$
6.1	$\{c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\{2\}$
4.2	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$
5.2	$\{c\}$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$
6.2	$\{c\}$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$

t	D_1	D_2	D_3	D_4	D_5	X'
0	$\{c\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{d\}$	$\emptyset$
1	$\{c\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{d\}$	$\emptyset$
2	$\{c, d\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{d\}$	$\emptyset$
3	$\{d\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{d\}$	$\emptyset$
4.1	$\{d\}$		$\{d\}$	$\{a\}$	$\{d\}$	$\{c\}$
5.1	$\{d\}$		$\{b, d\}$	$\{a\}$	$\{d\}$	$\{c\}$
6.1	$\{o, d\}$		$\{b\}$	$\{a\}$	$\{d\}$	$\{c\}$
4.2	$\{d\}$	$\{a, b\}$		$\{a\}$	$\{d\}$	$\{c\}$
5.2	$\{d\}$	$\{a, b\}$		$\{a\}$	$\{d\}$	$\{c\}$
6.2	$\{o, d\}$	$\{a, b\}$		$\{a\}$	$\{d\}$	$\{c\}$

5 Expected profits, Prices, and Strategical Issues

Since the history of MAPR is nondeterministic, we need to introduce concepts of buyers' *expected profits* and items' *expected prices*. Let R_*^t be a rationing system s.t. $R_*^t(i, a) = 1$ if $\{i, a\} \in M^t$ or $a \notin \bigcup_{e \in M^t} e$, and 0 otherwise. Because we can induce M^t from R_*^t. So M^t can be written as $M^{R_*^t}$. We say $\langle \mathbf{p}^t, R_*^t \rangle$ is an allocation situation. Assume that the computation of MODS algorithm and the selection of items in Step 8 in Algorithm 2 are deterministic, all the lots happening in MAPR are fair[2]. Then i's expected profit and a's expected price on $\langle \mathbf{p}, R \rangle$ (i.e., $u_i^*(\mathbf{p}, R)$ and $\mathbf{p}_a^*(\mathbf{p}, R)$) are:

$$u_i^*(\mathbf{p}, R) = \begin{cases} V_i(\mathbf{p}, R) & \text{if } X_{min} = \emptyset \\ u_i^*(\mathbf{p}', R) & \text{if } \overline{X} = \emptyset \\ \frac{\sum_{i' \in N'} u_i^*(\mathbf{p}, R_{i'})}{|N'|} & \text{otherwise} \end{cases}$$

[2] Suppose there are k buyers drawing lots for the right to buy item a. Then the lot is fair if each one of these buyers has $1/k$ chance of winning the lot.

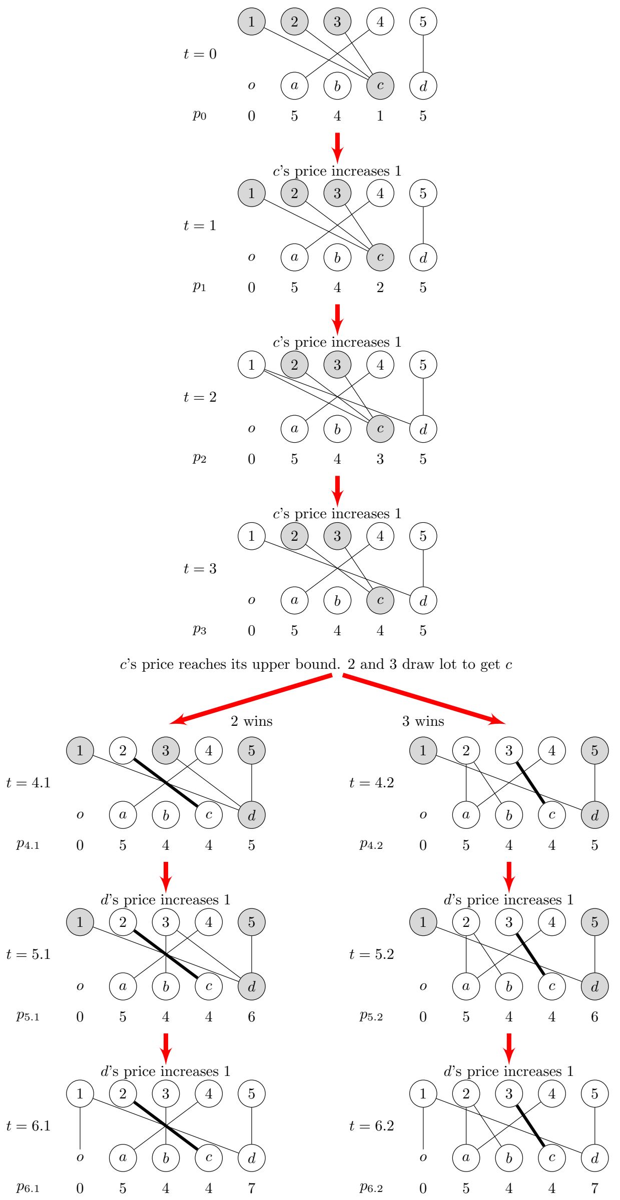

Fig. 4 Illustration of MAPR(step 1–7)'s execution

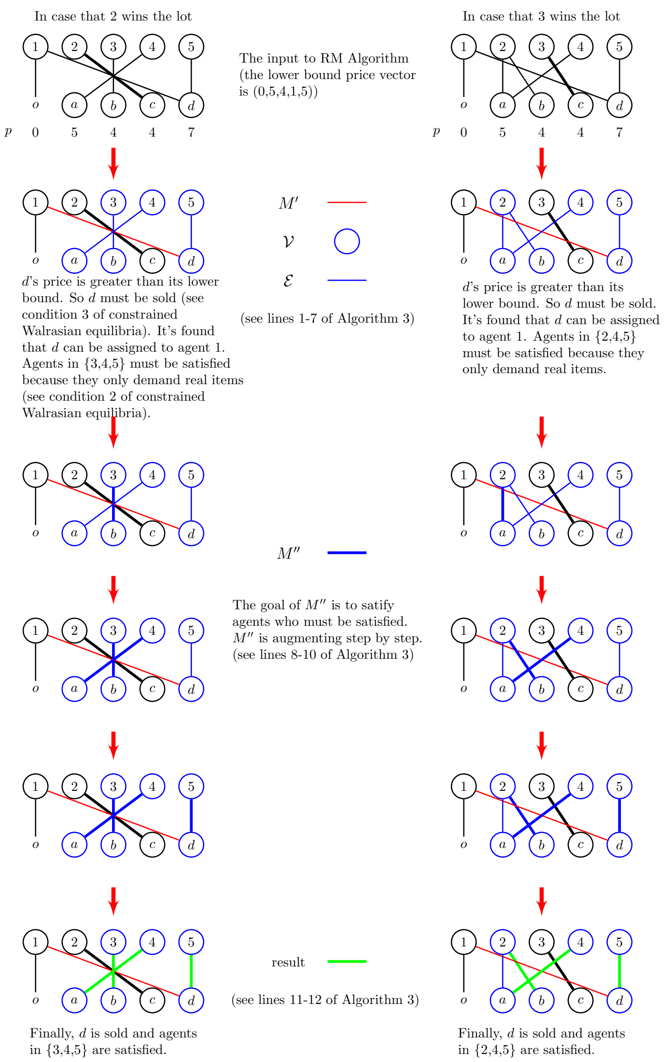

Fig. 5 Illustration of RM's execution

$$\mathbf{p}^*_a(\mathbf{p}, R) = \begin{cases} \mathbf{p}_a & \text{if } X_{min} = \emptyset \\ \mathbf{p}^*_a(\mathbf{p}', R) & \text{if } \overline{X} = \emptyset \\ \frac{\sum_{i' \in N'} \mathbf{p}^*_a(\mathbf{p}, R_{i'})}{|N'|} & \text{otherwise} \end{cases}$$

where (let $\mathcal{D} = (D_i(\mathbf{p}, R))_{i \in N}$):

- $X_{min} = \emptyset$ if $|\hat{M}_{\mathcal{D}}| = |\{i \in N | o \notin D_i(\mathbf{p}, R)\}|$, and MODS$(\mathcal{D}, \hat{M}_{\mathcal{D}})$ otherwise; $\overline{X} = \{a \in X_{min} | \mathbf{p}_a = \overline{\mathbf{p}}_a\}$;
- $\mathbf{p}'_a = \mathbf{p}_a$, $\forall a \notin X_{min}$ and $\mathbf{p}'_a = \mathbf{p}_a + 1$, $\forall a \in X_{min}$;
- $b \in \overline{X}$ is the item selected by the seller in Step 8 in Algorithm 2;
- $N' = \{i \in N | b \in D_i(\mathbf{p}, R) \subseteq X_{min}\}$;
- $\forall \langle i, a \rangle \in N \times X$: $R_{i'}(i, a) = R(i, a)$ if $a \neq b$; $R_{i'}(i, b) = 0$ if $i \neq i'$; and $R_{i'}(i', b) = 1$.

In fact, $u^*_i(\mathbf{p}, R)$ and $\mathbf{p}^*_a(\mathbf{p}, R)$ can be computed by developing a search tree: each node is an allocation situation, and is expanded (if $X_{min} \neq \emptyset$) into (i) one single branch if $\overline{X} = \emptyset$, and (ii) $|N'|$ branches otherwise. See Tables 2 and 3. We can find that $u^*_1(\mathbf{p}^0, R^0_*) = 0.5 * u^*_1(\mathbf{p}^{6.1}, R^{6.1}_*) + 0.5 * u^*_1(\mathbf{p}^{6.2}, R^{6.2}_*) = 0$, $u^*_3(\mathbf{p}^0, R^0_*) = 0.5 * u^*_3(\mathbf{p}^{6.1}, R^{6.1}_*) + 0.5 * u^*_3(\mathbf{p}^{6.2}, R^{6.2}_*) = 2.5$, $\mathbf{p}^*_a(\mathbf{p}^0, R^0_*) = 0.5 * \mathbf{p}^*_a(\mathbf{p}^{6.1}, R^{6.1}_*) + 0.5 * \mathbf{p}^*_a(\mathbf{p}^{6.2}, R^{6.2}_*) = 5$.

As most collective decision mechanisms, MAPR is generally not *strategyproof* (in the sense of expected profit). For instance, see Example 4. If buyer 1 reports her demands sincerely, then her expected profit is 0. However, if 1 knows other buyers' valuations and reports strategically, then she reports $\{c\}$ from $t = 0$ to $t = 3$ (i.e., as if her valuation to item c is not less than 7), then reports sincerely, then her expected profit changes to 1/3, which makes her better off.

Now we are interested in two questions: (1) is MAPR strategyproof for some restricted domains? (2) when it is not, how hard is it for an buyer who knows the valuations of the others to compute an optimal strategy?

First we define reporting strategies and manipulation problems formally. Without loss of generality, let 1 be the manipulator. Note that not every sequence of 1's demands is reasonable. For instance, see Example 4 and Table 3. The seller can detect 1's manipulation if 1 reports $\{c\}$, $\{c\}$, $\{c,d\}$, and $\{c\}$ at $t=0$, 1, 2, and 3, respectively, because there is no value function u s.t. $u(c)-\mathbf{p}_c^2=u(c)-3=u(d)-5=u(d)-\mathbf{p}_d^2=u(d)-\mathbf{p}_d^3<u(c)-\mathbf{p}_c^3=u(c)-4$. A *strategy* for buyer 1 is a value function $u: X\rightarrow\mathbb{Z}_+$ with $u(o)=0$. So 1 can safely manipulate the process of MAPR when she reports her demands according to u completely (as if u is her true value function). A *manipulation problem* M (for buyer 1) is a 5-tuple $\langle N,X,\{u_i\}_{i\in N},\underline{\mathbf{p}},\overline{\mathbf{p}}\rangle$ where $\langle N,X,\{u_i\}_{i\in N}\rangle$ is an economy, $\underline{\mathbf{p}}$ and $\overline{\mathbf{p}}$ are the lower and upper bound price vectors on X, respectively. A strategy for M is *optimal* if 1 can not strictly increase her expected profit by reporting her demands according to any other strategy.

Now, back to Question (1): we show that the answer is positive when there are two buyers.

Theorem 3 *Let $N=\{1,2\}$ and $M=\langle N,X,\{u_i\}_{i\in N},\underline{\mathbf{p}},\overline{\mathbf{p}}\rangle$ be a manipulation problem. Then u_1 is optimal for M.*

Proof. Suppose that if 1 reports sincerely, then her expected profit is Δ. Let D_1 and D_2 be 1 and 2's true demands at $\underline{\mathbf{p}}$ and R respectively, where $R(i,a)=1$ for each $i\in N$ and $a\in X$.

Obviously, if $D_1\cup D_2=\{o\}$ or $|D_1\cup D_2|\geq 2$ (i.e., $X_{min}=\emptyset$ at $t=0$) then $\Delta=\max_{a\in X}(u_1(a)-\underline{\mathbf{p}}_a)$, which is the best possible outcome for 1. So u_1 is optimal in these cases.

Now, suppose $D_1=D_2=\{a\}$ s.t. $a\neq o$. Pick any strategy u'. Let $k=\overline{\mathbf{p}}_a-\underline{\mathbf{p}}_a$, $k_i=u_i(a)-\underline{\mathbf{p}}_a-\max_{b\in X\setminus\{a\}}(u_i(b)-\underline{\mathbf{p}}_b)$, $b_i\in X\setminus\{a\}$ s.t. $u_i(b_i)-\underline{\mathbf{p}}_{b_i}=u_i(a)-\underline{\mathbf{p}}_a-k_i$, and $\hat{k}=\min(k,k_1-1,k_2-1)$. Then if 1 applies strategy u_1, then she will report D_1 from $t=0$ to $t=\hat{k}$ and:

1. if $\hat{k}=k$, then $\Delta=0.5*(u_1(a)-\underline{\mathbf{p}}_a-k)+0.5*(u_1(b_1)-\underline{\mathbf{p}}_{b_1})=u_1(b_1)-\underline{\mathbf{p}}_{b_1}+0.5*(k_1-k)>u_1(b_1)-\underline{\mathbf{p}}_{b_1}$. If 1 applies u' instead, then her expected profit will not be better than $u_1(b_1)-\underline{\mathbf{p}}_{b_1}<\Delta$ if $u'(a)-\underline{\mathbf{p}}_a-\max_{b\in X\setminus\{a\}}(u'(b)-\underline{\mathbf{p}}_b)\leq k$, and will not be better than Δ otherwise.
2. if $k>\hat{k}=k_1-1$, then $\Delta=u_1(b_1)-\underline{\mathbf{p}}_{b_1}$. Because 2 can insist on $\{a\}$ to $t=\min(k,k_2-1)\geq k_1-1$, 1's expected profit can not be better than Δ.
3. if $k>\hat{k}=k_2-1$, then $\Delta=u_1(a)-\underline{\mathbf{p}}_a-k_2\geq u_1(a)-\underline{\mathbf{p}}_a-k_1=u_1(b_1)-\underline{\mathbf{p}}_{b_1}$. Because 2 can insist on $\{a\}$ to $t=k_2-1$, 1's expected profit can not be better than Δ.

So in all cases, 1 can not strictly increase her expected profit by applying strategy u'. Then u_1 is optimal for M. □

6 Conclusions and Future Works

We have presented a decentralized protocol (dynamic mechanism) for allocating indivisible resources under price rigidities, and proved formally that it can discover constrained Walrasian equilibria in polynomial time. We also have investigated the protocol from the points of computation of buyers' expected profits and items' expected prices, and discussed the manipulation (by one buyer) problem in the sense of buyer's expected profit.

Future work includes proving the conjecture about the complexity of manipulation (in the sense of expected profits) by one buyer, studying manipulation (in the sense of expected prices) by one or more buyers (whose manipulation motivation is not to buy some resources but to put up the prices of some resources), studying the problems of allocating divisible resources under prices rigidities. The last and most important future work is to apply our methodology to allocate some public resources (e.g., public housing, residential parking spaces, etc.) in real life.

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